Calculus of residues. Rouche Theorem

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Let Ω be a region, $z_0 \in \Omega$, $f \in \mathcal{A}(\Omega \setminus \{z_0\})$

<u>Def</u>. The <u>residue</u> of f at z_0 is defined as

$$R := \operatorname{Res}_{z=z_0} f(z) := \frac{1}{2\pi i} \oint_{C_r} f(z)\,dz,$$ where C_r is $\{z_0 + re^{it}\}$, the circle of radius r, centered at z_0 oriented counterclockwise.

$r < \operatorname{dist}(z_0, \partial\Omega)$.

<u>Remark</u>. It <u>does not</u> depend on r.

Indeed, if $r \neq r'$, $\gamma := C_r - C_{r'} \sim 0$ in $\Omega \setminus \{z_0\}$:  : $n(C_r, z_0) - n(C_{r'}, z_0) = 0$

$\forall z \notin \Omega \Rightarrow z$ is in the unbounded component of $\mathbb{C} \setminus \gamma$.

<u>Theorem</u>. $R = \operatorname{Res}_{z=z_0} f(z)$ is the unique number such that

$$f(z) - \frac{R}{z - z_0} \quad \text{has antiderivative in } B(z_0, \operatorname{dist}(z_0, \partial\Omega)) \setminus \{z_0\}.$$

<u>Proof</u>. (Uniqueness)

$f(z) - \frac{R}{z-z_0}$, $f(z) - \frac{R'}{z-z_0}$ has antiderivative $\Rightarrow$ $f(z) - \frac{R}{z-z_0} - \left(f(z) - \frac{R'}{z-z_0}\right) = \frac{R'-R}{z-z_0}$ has antiderivative $\Rightarrow$

$$\Rightarrow 0 = \oint_{C_r} \frac{R-R'}{z-z_0}\,dz = 2\pi i (R - R') \Rightarrow R = R'.$$

(Existence).

$\gamma \subset B(z_0, \operatorname{dist}(z_0, \partial\Omega)) \setminus \{z_0\}$

Then $\gamma \sim 0$ in $B(z_0, \operatorname{dist}(z_0, \partial\Omega))$

$C_r \sim 0$ "

Let $\gamma' := \gamma - n(\gamma, z_0) C_r$. $R = \operatorname{Res}_{z=z_0} f(z)$

Then $n(\gamma', z_0) = n(\gamma, z_0) - n(\gamma, z_0) = 0$, so $\gamma' \sim 0$ in $B(z_0, \operatorname{dist}(z_0, \partial\Omega)) \setminus \{z_0\}$.

So $0 = \oint_{\gamma'} f(z)\,dz = \oint_\gamma f(z)\,dz - n(\gamma, z_0) \oint_{C_r} f(z)\,dz =$

$$\oint_\gamma f(z)\,dz - n(\gamma, z_0)\, R \cdot 2\pi i = \oint_\gamma f(z)\,dz - \oint_\gamma \frac{R\,dz}{z - z_0} = \oint_\gamma \left(f(z) - \frac{R}{z - z_0}\right) dz.$$

$$\oint_\gamma f(z)\,dz - n(\gamma,z_0)\,R\cdot 2\pi i = \oint_\gamma f(z)\,dz - \oint_\gamma \frac{R\,dz}{z-z_0} = \oint_\gamma \left(f(z) - \frac{R}{z-z_0}\right)dz.$$

$$n(\gamma,z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{dz}{z-z_0}$$

<u>Theorem</u> (Residue Theorem).

Let Ω be a region, $I\subset\Omega$ - a discrete set (i.e. $\forall z\in\Omega\ \exists\,\delta>0 : (B(z,\delta)\setminus\{z\})\cap I=\emptyset$).

Let $\gamma\subset\Omega$ - a cycle, $\gamma\cap I=\emptyset$, $\gamma\sim 0$ in Ω.

Let $f\in A(\Omega\setminus I)$.

<u>Then</u>: $\frac{1}{2\pi i}\oint_\gamma f(z)\,dz = \sum_{w\in I} n(\gamma,w)\,\mathrm{Res}_{z=w} f(z)$

<u>Remark</u>. The set $I_{n_\gamma}=\{w : n(\gamma,w)\neq 0\}$ is bounded and $\subset\Omega$. Thus $I\cap I_{n_\gamma}$ is finite (otherwise, $\exists\, z_j\in I\cap I_{n_\gamma}$, $z_j\to z\in\Omega$. z does not satisfy the discreteness). So the sum on RHS is finite! ∎

<u>Proof</u>. Let $w_1,\dots,w_K$ be singularities in I_{n_γ}.

Let us choose $r_k>0$: 1) $r_k<\mathrm{dist}(w_k,\gamma)$

2) $r_k<\frac{|w_j-w_k|}{4}$ $\forall\, j\neq k$.

Let $C_k:=\{w_k+r_ke^{it}\}$.

Then, as before: $\gamma' := \gamma-\sum n(\gamma,w_k)C_k\sim 0$ <u>in $\Omega\setminus I$</u>.

(because $n(\gamma-\sum n(\gamma,w_k)C_k, w_j) = n(\gamma - n(\gamma,w_j)C_j, w_j) =$

$n(\gamma,w_j)-n(\gamma,w_j)=0$).

So $\oint_{\gamma'} f(z)\,dz=0 \Rightarrow \oint_\gamma f(z)\,dz = \sum_{k=1}^{n} n(\gamma,w_k)\underbrace{\oint_{C_k} f(z)\,dz}_{2\pi i\,\mathrm{Res}_{z=w_k} f(z)}$ ∎

__Corollary.__ Let γ be the oriented boundary of a region Ω. $\{z_1, \dots, z_n\} \subset \Omega$ – finite set, $f \in A((\Omega \cup \gamma) \setminus I)$. Then

$$\frac{1}{2\pi i} \oint_\gamma f(z)\,dz = \sum_{k=1}^{n} \operatorname{Res}_{z=z_k} f(z).$$

__How to calculate residues?__

__Case 1.__ Simple pole at z_0.

$$f(z) = \frac{g(z)}{z-z_0}, \quad g \in A(B(z_0,r)),\ g(z_0) \neq 0.$$

$$\operatorname{Res}_{z=z_0} f(z) = \frac{1}{2\pi i} \oint_{C_r} f(z)\,dz = \frac{1}{2\pi i} \oint_{C_r} \frac{g(z)}{z-z_0}\,dz = g(z_0) = \boxed{\lim_{z\to z_0} (z-z_0) f(z)}$$

And if $f \in A(B(z_0,r) \setminus \{z_0\})$, and $\exists \lim_{z\to z_0} (z-z_0) f(z) =: R$, then $g(z) = \begin{cases} f(z)(z-z_0), & z \neq z_0 \\ R, & z = z_0 \end{cases} \in A(B(z_0,r))$. So $R = \operatorname{Res}_{z=z_0} f(z)$, as above.

__Example.__ $\operatorname{Res}_{z=0} \frac{1}{\sin z} = \lim_{z\to 0} \frac{z}{\sin z} = \frac{1}{\lim_{z\to 0} \frac{\sin z - \sin 0}{z - 0}} = \frac{1}{\cos 0} = 1.$

If $f(z) = \frac{g(z)}{h(z)}$, $g, h \in A(B(z_0,r))$, $h(z_0) = 0$, $h'(z_0) \neq 0$ (simple __zero__).

Then $\operatorname{Res}_{z=z_0} f(z) = \lim_{z\to z_0} \frac{g(z)}{h(z)} \cdot (z-z_0) = \frac{\lim_{z\to z_0} g(z)}{\lim_{z\to z_0} \frac{h(z)-h(z_0)}{z-z_0}} = \boxed{\frac{g(z_0)}{h'(z_0)}}$

Example. Res ... $\frac{\cos 0}{\ldots} = 1$

$\ldots_{z=z_0} \quad \ldots_{z\to z_0} \quad \ldots \lim_{z\to z_0} \frac{h(z)-h(z_0)}{z-z_0} \quad h'(z_0)$

Example $\mathrm{Res}_{z=0} \cot z = \frac{\cos 0}{\sin' 0} = 1.$

Case 2. f has a pole of order h at z_0.

$$f(z) = \frac{g(z)}{(z-z_0)^h}, \text{ where } g(z_0) \neq 0, \quad g \in A(B(z_0, r)).$$

By Cauchy:

$$g^{(h-1)}(z_0) = (h-1)! \frac{1}{2\pi i} \oint_{C_\delta} \frac{g(z)}{(z-z_0)^h} dz = (h-1)! \frac{1}{2\pi i} \oint_{C_\delta} f(z) dz = \mathrm{Res}_{z=z_0} f(z) (h-1)!$$

So

$$\boxed{\mathrm{Res}_{z=z_0} f(z) = \frac{1}{(h-1)!} \left(\frac{d^{h-1}}{dz^{h-1}} \left((z-a)^h f(z) \right) \right).}$$

Case 3. Essential singularity: no good formula.
(There will be one later).

General argument principle.

Theorem. Let $f \in \mathcal{M}(\Omega)$. $\gamma \subset \Omega$ - cycle, $\gamma \sim 0$ in Ω.

$\forall z \in \gamma$, $f(z) \neq 0, \infty$. (no poles or zeroes on γ)

Then $$\frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)} dz = \sum_{f(z)=0} n(\gamma, z) \operatorname{ord}(f, z) + \sum_{f(z)=\infty} n(\gamma, z) \operatorname{ord}(f, z).$$

Reminder: "Argument principle" because

$$\frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)} dz = n(f \circ \gamma, 0).$$

Remark. As usual, there are only finitely many zeroes and poles of f for which $n(\gamma, z) \neq 0$, so both sums on RHS are finite.

Proof. $\frac{f'(z)}{f(z)} \in A(\Omega \setminus \{z : f(z) = 0 \text{ or } f(z) = \infty\})$

So, by residue theorem

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{\substack{f(w)=0\\ f(w)=\infty}} n(\gamma, z)\, \mathrm{Res}_{z=w} \frac{f'(z)}{f(z)}.$$

Let $\mathrm{ord}(f, w) = h \neq 0$ (w is a zero or pole).

Then $f(z) = (z-w)^h f_1(z)$, where $f_1(w) \neq 0$, $f_1 \in A(B(w, \delta))$ for some $\delta > 0$.

$$\frac{f'(z)}{f(z)} = \frac{h(z-w)^{h-1} f_1(z) + (z-w)^h f_1'(z)}{(z-w)^h f_1(z)} = \frac{h}{z-w} + \frac{f_1'(z)}{f_1(z)}$$

. (we also knew it by properties of logarithmic derivative).

So $\mathrm{Res}_{z=w} \frac{f'(z)}{f(z)} = \lim_{z \to w} (z-w)\left(\frac{h}{z-w} + \frac{f_1'(z)}{f_1(z)}\right) = h + 0 \cdot \frac{f_1'(w)}{f_1(w)} = h = \mathrm{ord}(f, w)$. ∎

Corollary Let γ be an oriented boundary of Ω, $f \in \mathcal{M}(\Omega \cup \gamma)$, $f(z) \neq 0$ and $f(z) \neq \infty$ if $z \in \gamma$.

Then

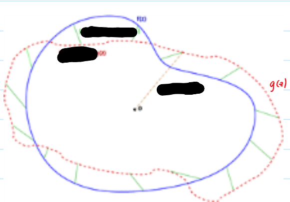
$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\,dz = N - P$$

, where N is the number of zeroes of f in Ω (counted with multiplicity), P – the number of poles in Ω (also counted with multiplicity).

Eugène Rouché

Theorem (Rouché) Let $f, g \in A(\Omega)$, γ - simple closed curve in Ω, $\gamma \sim 0$ in Ω. Assume
$\forall z \in \gamma \quad |f(z) - g(z)| < |f(z)|$.
Then f and g have the same number of zeroes (N_f and N_g) inside of γ, counted with multiplicities.

Heuristics. We need to prove: $N(f \circ \gamma, 0) = N(g \circ \gamma, 0)$ (Argument principle)
But $g \circ \gamma(t)$ is always at distance $|f(\gamma(t)) - g(\gamma(t))|$ from $f(\gamma(t))$, which is less then distance from $f(\gamma(t))$ to 0.
So it winds around 0 the same number of times!



Proof.

Let $\varphi(z) = \frac{g(z)}{f(z)}$. So $\frac{\varphi'(z)}{\varphi(z)} = \frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)}$

On γ: $|\varphi(z) - 1| < 1$.

So $\varphi \circ \gamma \subset \{|\zeta - 1| \le 1\}$. $0 \notin \{|\zeta - 1| < 1\} \Rightarrow n(\varphi \circ \gamma, 0) = 0$.

But $0 = n(\varphi \circ \gamma, 0) = \frac{1}{2\pi i} \oint_\gamma \frac{\varphi'(z)}{\varphi(z)} dz = \frac{1}{2\pi i}\left(\oint_\gamma \frac{g'(z)}{g(z)} dz - \oint_\gamma \frac{f'(z)}{f(z)} dz\right) =$

$N_g - N_f$. ■

Yet another proof of FTA.

Let $p(z) = a_d z^d + \underbrace{a_{d-1} z^{d-1} + \dots + a_0}_{q(z)}$

Let $f(z) = a_d z^d$.

Then $\lim_{z \to \infty} \frac{q(z)}{f(z)} = \sum_{k=0}^{d-1} \frac{a_k}{a_d} \lim_{z \to \infty} z^{k-d} = 0$.

So for large R, if $|z| = R$ then $\frac{|p(z) - f(z)|}{|f(z)|} = \frac{|q(z)|}{|f(z)|} < 1$.

So, by Rouche applied to $C_R = \{Re^{it}\}$,

$N_p = N_f = d$ ■

Adolf Hurwitz

Theorem (Hurwitz).

Assume $f_n \in A(\Omega)$, $\forall z \in \Omega$ $f_n(z) \neq 0$. Let $f_n \to f$ locally uniformly.

Then $\forall z \in \Omega$ $f(z) \neq 0$ or $f(z) \equiv 0$.

Proof. By Weierstrass Theorem, $f \in A(\Omega)$.

~~Assume~~: $f \not\equiv 0$, $f(z_0) = 0$. Then $\exists r > 0$:

$0 < |z - z_0| \leq r \Rightarrow z \in \Omega$, $f(z) \neq 0$ (zeroes are isolated).

Let $C_r = \{ |z - z_0| = r \}$. Then $m = \min\limits_{z \in C_r} |f(z)| > 0$.

$f_n \rightrightarrows f$ uniformly on C_r. Take k: $|f_k(z) - f(z)| < m$ $\forall z \in C_r$.

Then, by Rouche, f_k and f have the same number of zeroes inside C_r ($|f_k(z) - f(z)| < m \leq |f(z)|$). But $f_k(z) \neq 0 \, \forall z$, $f(z_0) = 0$ – contradiction! ∎

Corollary. $f_n \in A(\Omega)$, injective (= conformal).

$f_n \to f$ locally uniformly on Ω. Then either f is conformal or $f \equiv$ const.

Proof

Assume $f \not\equiv$ const.

Fix $z_0 \in \Omega$. Consider $g_n(z) := f_n(z) - f_n(z_0) \neq 0$ in $\Omega \setminus \{z_0\}$.

$g_n(z) \to f(z) - f(z_0)$ locally uniformly, $g_n(z) \neq 0$ in $\Omega \setminus \{z_0\}$.

So, by Hurwitz, $f(z) \neq f(z_0)$ $\forall z \in \Omega \setminus \{z_0\}$.

So for $z \neq z_0$, $f(z) \neq f(z_0)$ – injective ∎